Mr.Arup Kumar Majee, Principal,
Jnan Chandra Ghosh Polytechnic

Chapter 1

Alternating Current Circuits

1.1 Generation of AC Waveform

A coil of single turn having length of l m and width of b m is placed within a magnetic field of flux density B Wb/m^2 as shown in Figure 1.1. It is rotated at an angular speed of ω rad/sec. Speed of the coil, $n_s = \frac{\omega}{2\pi}$

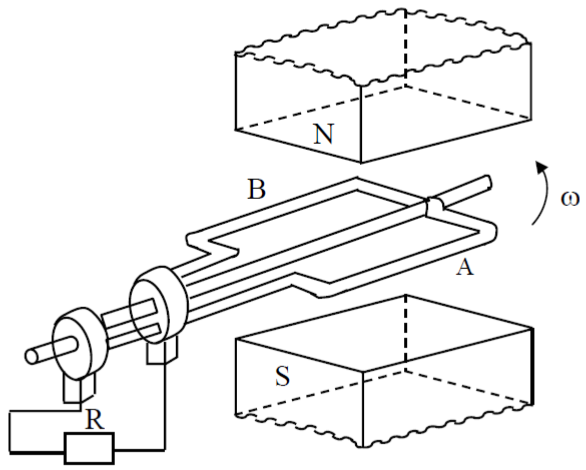


Figure 1.1: Generation of AC in a single turn

rev/sec.

For a two pole machine, $n_s = f$

Tangential velocity of the coil $u = \frac{b}{2}\omega = \pi b n_s = \pi b f$

At a certain instant when the coil is at an angle $\theta = \omega t$,emf induced in any one side of the coil

$$e = Blu\sin\theta$$

Emf induced in the coil (considering both sides A and B)

$$e = 2Blu\sin\theta$$

Emf induced in a coil of N number of turns

$$e = 2BlNu\sin\theta = 2Bl\pi bfN\sin\theta = E_m\sin\omega t$$

$$E_m = 2\pi fB(lb)N = 2\pi f\phi_m N$$

where area of the coil, $a = lb$ m^2 and $\phi_m = B.a$ is the maximum flux passing through the coil when it is in horizontal position.

1.2 Difference between a.c. & d.c.

- Alternating current is generated by a.c. generators.
  Direct current is generated by d.c. generator or battery.

- Alternating current varies both in magnitude and direction over a cycle.
  Direct current is unidirectional and usually constant in magnitude.
- Alternating current is opposed by circuit impedance.
  Direct current gets opposed by only resistance.
- Alternating current has 50Hz frequency in India and power factor depends on the circuit parameters.
  Direct current has zero frequency and power factor is always 1.

1.3 Definitions

Cycles: The pattern of a sine wave (from zero to a positive peak to zero to a negative peak and back to zero) repeats after every time period (T) or 2π radians. One complete set of positive and negative values of the function is called a **cycle**.

Frequency: The number of cycles completed in one second is called the frequency of an alternating quantity. It is measured in hertz (Hz).

$$1Hz = \text{ 1 cycles/ second}$$

Time period is the time taken by an alternating wave to complete one cycle. It is measured in seconds.

Instantaneous value: The value of alternating quantity at a given instant of time is called the instantaneous value. It varies from instant to instant. They are denoted by lower case letters.

Maximum value: This is the maximum value attained by the alternating quantity in a complete cycle. This is the highest among all the instantaneous values. This is also known as peak value, crest value or amplitude.

Example 1.1 *The maximum value of a sinusoidal alternating current of frequency 50Hz is* $50A$. *Write the expression for instantaneous value of the alternating current. Compute the value of current at 3ms and 15ms.*

Solution

$$i = 50\sin(2 \times pi \times 50t) = 50\sin 314.1593t$$
$$\text{At t=3ms} i = 40.4508A$$
$$\text{At t=15ms} i = -50A$$

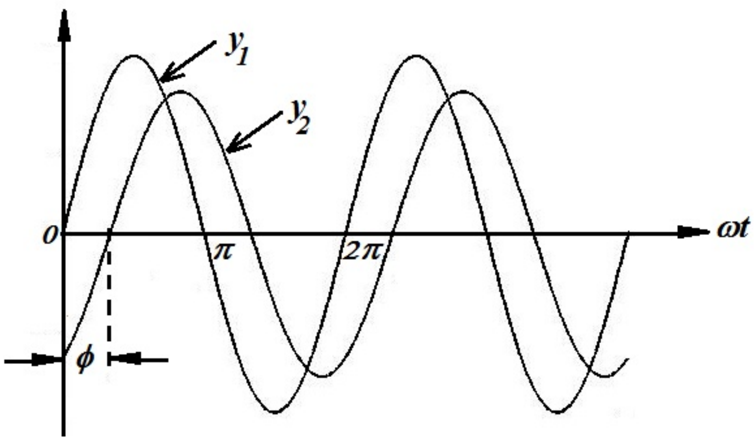


Figure 1.2: Illustration of phase difference

Phase It is the fraction of a cycle or time period that has been elapsed since it has last passed the chosen origin. The phase of a wave at time t is expressed as t/T.

Phase angle It is the phase expressed in radians or degrees, denoted by θ.

$$\theta = \frac{2\pi t}{T} \text{ radians}$$

Phase difference It is the angular difference between two sinusoidal waveforms, usually denoted by ϕ. If two sinusoidal waveforms shown in Figure 1.2 are expressed as:

$$\begin{aligned} y_1 &= Y_{m1}\sin\omega t \\ y_2 &= Y_{m2}\sin(\omega t-\phi) \end{aligned}$$

It may be observed that y_2 passes its zero value at $\omega t = \phi$ whereas y_1 passes its zero value at origin. Thus, it may be stated that y_2 lags y_1 by an angle ϕ or, in other words, y_1 leads y_2 by an angle ϕ. The angle, ϕ, is called the phase difference between y_1 and y_2.

1.3.1 Average Value

Average value of a given waveform may be derived from the following expression

$$I_{av} = \frac{\text{area under the curve}}{\text{length of the base of the curve}}$$

For any periodical function having a wave symmetrical about the horizontal axis, average value over a complete cycle is zero.

For a sine wave having amplitude I_m and angular frequency ω, half cycle average is given by

$$\begin{aligned} I_{av} &= \frac{1}{\pi}\int_0^{\pi} I_m\sin\theta d\theta \qquad \text{where } \theta=\omega t \\ &= \frac{I_m}{\pi}\left[-\cos\theta\right]_0^{\pi} \\ &= \frac{I_m}{\pi}\left[\cos 0-\cos\pi\right] \\ &= \frac{2I_m}{\pi} \end{aligned}$$

1.3.2 Root Mean Square Value

Root Mean Square Value: The effective value or r.m.s. value of an alternating current is equal to that value of direct current which produces the same heat in the same time in the same resistor. Thus effective value is the d.c. equivalent of the a.c.

$$I_{rms} = \sqrt{\frac{1}{\pi}\int_0^{\pi} i^2 d\theta}$$

For a sinusoidal wave having amplitude I_m and angular frequency ω

$$\begin{aligned} I_{rms} &= \sqrt{\frac{1}{\pi}\int_0^{\pi} I_m^2\sin^2\theta d\theta} \qquad \text{where } \theta=\omega t \\ &= \sqrt{\frac{I_m^2}{2\pi}\int_0^{\pi}(1-\cos 2\theta)d\theta} \\ &= \sqrt{\frac{I_m^2}{2\pi}\left[\theta-\frac{1}{2}\sin 2\theta d\theta\right]_0^{\pi}} \\ &= \sqrt{\frac{I_m^2}{2\pi}\times\pi} = \frac{I_m}{\sqrt{2}} \end{aligned}$$

Example 1.2 *An ac voltage is expressed as $v = 282.84\sin(157.08t+\pi/2)volts$. Determine its (a) average value over half cycle, (b) rms value, (c) frequency, (d) periodic time and (e) form factor.*

Solution

$$v = 282.84\sin(314.16t + \pi/2)\ volts$$

(a) average value over half cycle, $V_{av} = \frac{2 \times V_m}{\pi} = \frac{2 \times 282.84}{\pi} = 180.0615\ volts$

(b) rms value, $V_{rms} = \frac{V_m}{\sqrt{2}} = \frac{282.14}{\sqrt{2}} = 199.9981\ volts$

(c) frequency, $f = \frac{\omega}{2\pi} = \frac{314.16}{2\pi} = 50\ Hz$

(d) periodic time, $T = \frac{1}{f} = 0.02\ sec$

(e) form factor, $k_f = \frac{199.9981}{180.0615} = 1.1107$

Example 1.3 *An alternating current is expressed as $i = 124.70\sin 323t$. Determine its (a) amplitude, (b)frequency, (c) rms value, (d) average value over half cycle and (e) form factor.*

Solution

$$i = 124.70\sin 323t\ amps$$

(a) amplitude, $I_m = \frac{1}{f} = 124.7\ amps$

(b) frequency, $f = \frac{\omega}{2\pi} = \frac{323}{2\pi} = 51.407\ Hz$

(c) rms value, $I_{rms} = \frac{I_m}{\sqrt{2}} = \frac{282.14}{\sqrt{2}} = 88.1762\ amps$

(d) average value over half cycle, $I_{av} = \frac{2 \times I_m}{\pi} = \frac{2 \times 124.7}{\pi} = 79.3865\ amps$

(e) form factor, $k_f = \frac{I_{rms}}{I_{av}} = \frac{88.1762}{79.3865} = 1.1107$

Example 1.4 *Find rms value of the current given by $i = 10 + 5\cos(628t + 30^o)$.*

Solution

$$i = 10 + 5\cos(628t + 30^o)$$

Rms value of the current, $I_{rms} = \sqrt{10^2 + (5/sqrt2) * 2} = 10.6066A$

Example 1.5 *Compute the effective value of a resultant current in a wire which carries a direct current of 4A and a sinusoidal alternating current with an amplitude of 3A.*

Solution effective value of a resultant current,

$$\sqrt{\left(\frac{400}{\sqrt{2}}\right)^2 + \left(\frac{200}{\sqrt{2}}\right)^2 + \left(\frac{100}{\sqrt{2}}\right)^2} = 324.037A$$

Form factor for a particular waveform is defined as the ratio of the r.m.s. value to the average value. It is expressed as

$$k_f = \frac{\text{r.m.s. value}}{\text{average value}}$$

For a sinusoidal current, $i = I_m \sin\omega t$ (I_m is the peak value and ω is the anugular frequency in rad/sec)

$$k_f = \frac{I_m/\sqrt{2}}{2I_m/\pi} = \frac{\pi}{2\sqrt{2}} = 1.1107$$

Peak factor for a particular waveform is defined as the ratio of the peak (or maximum) value to the r.m.s. value. It is expressed as

$$k_p = \frac{\text{Peak value}}{\text{r.m.s. value}}$$

This is also known as *crest* factor or *amplitude* factor.

For a sine wave

$$k_p = \frac{I_m}{I_m/\sqrt{2}} = \sqrt{2} = 1.414$$

Example 1.6 *A certain waveform has a form factor of 1.2 and a peak factor of 1.5. If the maximum value is 100, find the rms and average values.*

Solution

$$
\begin{aligned}
k_f &= \frac{\text{rms value}}{\text{average value}} = \frac{Y_{rms}}{Y_{av}} = 1.2 \\
k_p &= \frac{\text{peak value}}{\text{rms value}} = \frac{Y_m}{Y_{rms}} = 1.5 \\
\text{rms value, } Y_{rms} &= \frac{Y_m}{k_p} = \frac{100}{1.5} = 66.6667 \\
\text{average value, } Y_{av} &= \frac{Y_{rms}}{k_f} = \frac{66.6667}{1.2} = 55.5556
\end{aligned}
$$

Example 1.7 *The waveform of a periodical voltage has a form factor of 1.15 and a peak factor of 1.4. If the peak value of voltage is 322 volts, compute (a) average value over half cycle and (b) rms value of this voltage.*

Solution

Peak value of voltage, $V_m = 322V$

Form factor, $k_f = \dfrac{V_{rms}}{V_{av}} = 1.15$

Peak factor, $k_p = \dfrac{V_m}{V_{rms}} = 1.4$

(b) Rms value over half cycle of the voltage, $V_{rms} = 322/1.4 = 230V$

(a) Average value over half cycle of the voltage, $V_{av} = 230/1.15 = 200V$

Example 1.8 *Determine average and rms value of a rectangular shaped current having an amplitude I_m and time period T. Also determine form factor and peak factor for this current.*

Solution The expression for this current $i = I_m$ for $0 < t < T/2$

$$
\begin{aligned}
I_{av} &= \frac{1}{T/2}\int_0^{T/2} I_m dt = \frac{2I_m}{T}\left[t\right]_0^{T/2} = I_m \\
I_{rms}^2 &= \frac{1}{T/2}\int_0^{T/2} I_m^2 dt = \frac{2I_m^2}{T}\left[t\right]_0^{T/2} = I_m^2 \\
I_{rms} &= I_m \\
k_f &= \frac{\text{rms value}}{\text{average value}} = \frac{I_{rms}}{I_{av}} = 1.0 \\
k_p &= \frac{\text{peak value}}{\text{rms value}} = \frac{I_m}{I_{rms}} = 1.0
\end{aligned}
$$

Example 1.9 *Determine average and rms value of a saw-tooth shaped current having an amplitude I_m and time period T. Also determine form factor and peak factor for this current.*

Solution The expression for this current $i = \left(\frac{I_m}{T}\right) t$ for $0 < t < T$

$$
\begin{aligned}
I_{av} &= \frac{1}{T}\int_0^{T} \left(\frac{I_m}{T}\right) t dt = \frac{I_m}{T^2}\left[\frac{t^2}{2}\right]_0^{T} = \frac{I_m}{2} \\
I_{rms}^2 &= \frac{1}{T}\int_0^{T} \left(\frac{I_m}{T} t\right)^2 dt = \frac{I_m^2}{T^3}\left[\frac{t^3}{3}\right]_0^{T} = \frac{I_m^2}{3} \\
I_{rms} &= \frac{I_m}{\sqrt{3}} \\
k_f &= \frac{\text{rms value}}{\text{average value}} = \frac{I_{rms}}{I_{av}} = \frac{2}{\sqrt{3}} = 1.1547 \\
k_p &= \frac{\text{peak value}}{\text{rms value}} = \frac{I_m}{I_{rms}} = \sqrt{3} = 1.732
\end{aligned}
$$

Example 1.10 *Determine average and rms value of a triangular shaped current having an amplitude I_m and time period T. Also determine form factor and peak factor for this current.*

Solution The expression for this current $i = \left(\frac{I_m}{T/4}\right) t$ for $0 < t < \frac{T}{4}$

$$I_{av} = \frac{1}{T/4}\int_0^{T/4}\left(\frac{I_m}{T/4}\right) tdt = \frac{16I_m}{T^2}\left[\frac{t^2}{2}\right]_0^{T/4} = \frac{I_m}{2}$$
$$I_{rms}^2 = \frac{1}{T/4}\int_0^{T/4}\left(\frac{I_m}{T/4}t\right)^2 dt = \frac{64I_m^2}{T^3}\left[\frac{t^3}{3}\right]_0^{T/4} = \frac{I_m^2}{3}$$
$$I_{rms} = \frac{I_m}{\sqrt{3}}$$
$$k_f = \frac{\text{rms value}}{\text{average value}} = \frac{I_{rms}}{I_{av}} = \frac{2}{\sqrt{3}} = 1.1547$$
$$k_p = \frac{\text{peak value}}{\text{rms value}} = \frac{I_m}{I_{rms}} = \sqrt{3} = 1.732$$

Example 1.11 *Find the relative heating effects of two in-phase current waveforms of equal peak values and time periods, but one sinusoidal and the other triangular.*

Solution Heating depends on the square of rms values.
For a sinusoidal current rms value $I_{sin} = \frac{I_m}{\sqrt{2}}$
For a triangular current rms value $I_{tr} = \frac{I_m}{\sqrt{3}}$
Relative heating effects

$$I_{sin}^2 : I_{tr}^2 = \frac{1}{2} : \frac{1}{3} = \frac{3}{2}$$

Example 1.12 *Find the room mean square value, average value and the form factor of the resultant current in a wire that carries a direct current of 5A and a sinusoidal alternating current with an amplitude of 5A.*

Solution

$$i = 5 + 5\sin\omega t = 5(1 + \sin\omega t)$$
$$I_{av} = \frac{1}{2\pi}\int_o^{2\pi} id(\omega t) = \frac{5}{2\pi}\left[(\omega t) + \cos\omega t\right]_o^{2\pi} = \frac{5}{2\pi}\left[(2\pi - 0) + (\cos 2\pi - cos0)\right] = 5A$$
$$I_{rms}^2 = \frac{1}{2\pi}\int_o^{2\pi} i^2 d(\omega t) = \frac{25}{2\pi}\int_o^{2\pi}\left[1 + 2\sin\omega t + \sin^2\omega t\right] d(\omega t)$$
$$= \frac{25}{2\pi}\int_o^{2\pi}\left[1 + 2\sin\omega t + \frac{1-\cos 2\omega t}{2}\right] d(\omega t)$$
$$= \frac{25}{2\pi}\int_o^{2\pi}\left[\frac{3}{2}(\omega t) + 2\cos\omega t + \frac{1}{4}\sin 2\omega t\right]_o^{2\pi}$$
$$= \frac{25}{2\pi}\left[\frac{3}{2}(2\pi - 0) + 2(\cos 2\pi - cos0) + \frac{1}{4}(\sin 4\pi - \sin 0)\right] = \frac{75}{2}$$
$$I_{rms} = 6.1237A$$
$$k_f = \frac{\text{rms value}}{\text{average value}} = \frac{I_{rms}}{I_{av}} = 1.2247$$

Alternative method:
Average value of $5\sin\omega t$ over a full cycle $= 0$
Average value of resultant current $= 5 + 0 = 5A$
Rms value of $5\sin\omega t$ over a full cycle $= \frac{5}{\sqrt{2}}$
Rms value of resultant current $= \sqrt{5^2 + \left(\frac{5}{\sqrt{2}}\right)^2} = 6.1237A$
Form factor, $k_f = \frac{\text{rms value}}{\text{average value}} = \frac{I_{rms}}{I_{av}} = 1.2247$

1.4 Concept of phasors

If a crank is rotated at an angular frequency ω in counter clockwise direction, a sinusoid may be obtained by vertically projecting the different positions of the crank with respect to its angular position at different instant time.

ChapterAC/picACfundamentals/picPhasor1.png

(a)

ChapterAC/picACfundamentals/picPhasor2.png

(b)

ChapterAC/picACfundamentals/picPhasor3.png

(c)

Figure 1.3: Generation of Sinusoidal wave by rotating a crank

Consider Figure ??, where the crank OA having a length of Y_m is in horizontal position. If it is rotated at a uniform angular speed ω, its tip traces points A, B, C, D. Its corresponding vertical components are plotted to obtain a sinusoidal wave as shown. Maximum value Y_m is obtained when the crank is in vertical position.

Figure ?? shows a sinusoidal wave which has a negative value at the chosen reference time $t = 0$. This is obtained when the crank is lagging the horizontal reference by an angle ϕ.

Figure ?? shows a sinusoidal wave which has a positive value at the chosen reference time $t = 0$. This is obtained when the crank is leading the horizontal reference by an angle ϕ.

Thus a sinusoidal wave may be represented by a rotating line having its magnitude equal to the amplitude of the sine wave. This rotating line is known as the **phasor**.

1.4.1 Phasor Diagram

ChapterAC/picACfundamentals/picPhaseDiff2.jpg

Figure 1.4: Illustration of phase difference using phasors

While dealing with a number of sinusoidal waves having same angular frequency, phasor diagram approach is very useful. It is very cumbersome to plot a number of sinusoidal waves. Instead of them we can simply plot some lines showing their phase relationship. This diagram is called the **phasor diagram**.

Figure ?? shows two sinusoidal waves expressed as

$$\begin{aligned} y_1 &= Y_{m1} \sin \omega t \\ y_2 &= Y_{m2} \sin(\omega t - \phi) \end{aligned}$$

We can simply represent them by two lines having length of Y_{m1} and Y_{m2} having an angle ϕ between them.

As the r.m.s. values of the sinusoidal waves are commonly used, we plot the phasor diagrams using their r.m.s. values Y_1 and Y_2 instead of their peak values.

Consider three sinusoidal waves expressed as

$$\begin{aligned} y_1 &= Y_{m1} \sin(\omega t + \phi_1) \\ y_2 &= Y_{m2} \sin \omega t \\ y_3 &= Y_{m3} \sin(\omega t - \phi_2) \end{aligned}$$

Any of these three phasors can be taken as **reference phasor**. Let us consider y_2 as the reference, as it starts its rotation from horizontal position. Phasor diagrams using their amplitudes and rms values are shown in Figures ?? and ?? respectively.

Points to remember

- Several sinusoidal quantities can be drawn on a phasor diagram only if they alternate at the same frequency.
- For phasor diagrams, counter clockwise direction is taken as positive.
- The phasor which has moved to counter clockwise direction from the reference phasor, it is said to be leading the reference.

ChapterAC/picACfundamentals/picPhasorD1.png ChapterAC/picACfundamentals/picPhasorD2.png

(a) using Amplitude

(b) using r.m.s. values

Figure 1.5: Phasor diagram of three Sinusoidal waves

- The phasor which has moved to clockwise direction from the reference phasor, it is said to be lagging the reference.
- In series circuits, same current flows through all the components, usually this current is taken as reference.
- In parallel circuits, same voltage is applied across all the components, usually this voltage is taken as reference.
- While plotting voltages and currents on the same phasor diagram, all voltages should be drawn using the same scale. Similarly all currents should be drawn using the same scale. But the scale for currents could be different from the scale used for the voltages.

1.4.2 Representation of phasors

Phasors are usually represented in two ways: polar form and cartesian form. In polar form, phasors drawn in Figure **??** are written as

$$\begin{aligned}
\overline{Y_1} &= Y_1\angle + \phi_1 \\
\overline{Y_2} &= Y_2\angle 0^o \\
\overline{Y_3} &= Y_3\angle - \phi_2
\end{aligned}$$

Positive sign of ϕ_1 represents Y_1 leads Y_2 by an angle ϕ_1 whereas negative sign of ϕ_2 represents Y_3 lags Y_2 by an angle ϕ_2.

Polar form of representation is helpful for multiplication and division of two or more phasors.

In cartesian form, phasors drawn in Figure **??** are written as

$$\begin{aligned}
\overline{Y_1} &= Y_1\left(\cos\phi_1 + \jmath\sin\phi_1\right) \\
\overline{Y_2} &= Y_2(1 + \jmath 0) \\
\overline{Y_3} &= Y_3\left(\cos\phi_2 - \jmath\sin\phi_2\right)
\end{aligned}$$

where $j = \sqrt{-1}$.

Positive sign of $\sin\phi_1$ represents Y_1 leads Y_2 by an angle ϕ_1 whereas negative sign of $\sin\phi_2$ represents Y_3 lags Y_2 by an angle ϕ_2. Cosine component is known as real part whereas sine component is called imaginary part.

Cartesian form of representation is helpful for addition and subtraction of two or more phasors.

j operator

This is a operator frequently used by Electrical Engineers. When operated on a phasor, it helps to rotate a phasor by 90^o in anti-clockwise direction without changing its magnitude. Numerical value of $\jmath$-operator is $\sqrt{-1}$ and $\jmath^2 = -1$.

ChapterAC/picACfundamentals/picJOperator.png

Figure 1.6: Illustration of function of $\jmath$-operator

Let us consider OA as a phasor $\overline{Y} = Y\angle 0^o$ (shown in Figure **??**). When $\jmath$ operator is applied to it, it takes position OB. When $\jmath$ operator is applied to $\jmath Y$, it takes position OC. When $\jmath$ operator is applied to $\jmath^2 Y$, it takes position OD. When $\jmath$ operator is applied to $\jmath^3 Y$, it returns to its original position OA.

$$\begin{aligned}
OA &= \overline{Y} = Y\angle 0^o \\
OB &= \jmath(\overline{Y}) = Y\angle 90^o \\
OC &= \jmath(\jmath\overline{Y}) = Y\angle 180^o = -Y \\
OD &= \jmath(\jmath^2\overline{Y}) = Y\angle 270^o = Y\angle - 90^o \\
OA &= \jmath(\jmath^3\overline{Y}) = Y\angle 0^o
\end{aligned}$$

Hence, $\jmath = \sqrt{-1}$ causes 90^o counter clockwise rotation of a phasor.
$\jmath^2 = -1$ causes 180^o rotation of a phasor.
$\jmath^3 = \jmath \times \jmath^2 = -\sqrt{-1}$ causes 270^o counter clockwise rotation or 90^o clockwise rotation of a phasor.

1.4.3 Relationship between Polar and Cartesian representation

Consider a phasor, $\overline{A}$ shown in Figure **??**. This phasor, $\overline{A}$ in cartesian form may be written as

ChapterAC/picACfundamentals/picPhasor4.png

Figure 1.7: Representation of phasor

$$\overline{A} = a_x + \jmath a_y \tag{1.1}$$

This phasor, $\overline{A}$ in polar form may be written as

$$\overline{A} = A\angle\phi = A(\cos\phi + \jmath\sin\phi) \tag{1.2}$$

where A is the magnitude and ϕ is the phase angle of the phasor.
Comparing equations **??** and **??** we get

$$\begin{aligned} a_x &= A\cos\phi \\ a_y &= A\sin\phi \\ A &= \sqrt{a_x^2 + a_y^2} \\ \phi &= \tan^{-1}\frac{a_y}{a_x} \end{aligned}$$

Example 1.13 *An alternating current $I = (3 + \jmath 4)A$flows through a resistor of 40Ω. What will be the power consumed by the resistor?*

Solution

$$I = (3 + \jmath 4)A = 5A\angle 53.13^o$$

Power consumed by the resistor, $P = I^2R = 5^2 \times 40 = 1000W$

Example 1.14 *Convert the following phasors from rectangular form to polar form: (a) $A = 4 + \jmath 3$; (b) $B = 6 - \jmath 8$; (c) $C = -9 - \jmath 12$.*

Solution

$$\begin{aligned} (a)A &= 4 + \jmath 3 = 5\angle 36.8699^o \\ (b)B &= 6 - \jmath 8 = 10\angle 53.1301^o \\ (c)C &= -9 - \jmath 12 = 15\angle - 126.8699^o \end{aligned}$$

Example 1.15 *Convert the following phasors from polar form to rectangular form: (a) $A = 5\angle - 36.8699^o$; (b) $B = 10\angle 53.1301^o$; (c) $C = 10\angle - 150^o$.*

Solution

$$\begin{aligned} (a)A &= 5\angle - 36.8699^o = 4 - \jmath 3 \\ (b)B &= 10\angle 53.1301^o = 6 + \jmath 8 \\ (c)C &= 10\angle - 150^o = -8.6603 - \jmath 5 \end{aligned}$$

1.4.4 Phasor Algebra

Phasors are treated just like the vectors.

ChapterAC/picACfundamentals/picPhasorAdd.png

Figure 1.8: Phasor addition

Phasor Addition Consider two phasors A and B expressed as:

$$\begin{aligned}\overline{A} &= a_x + \jmath a_y \\ \overline{B} &= b_x + \jmath b_y \\ \overline{R} &= r_x + \jmath r_y = \bar{A} + \bar{B} = (a_x + b_x) + \jmath(a_y + b_y)\end{aligned}$$

Example 1.16 *In a parallel circuit, branch currents are given by $I_1 = (3 + \jmath 4)A$ and $I_2 = (4 + \jmath 3)A$. What will be the line current?*

Solution

$$\begin{aligned}I_1 &= (3 + \jmath 4)A \\ I_2 &= (4 + \jmath 3)A \\ I &= I_1 + I_2 = (3 + 4) + \jmath(4 + 3) = (7 + \jmath 7)A = 9.9A\angle 45^o\end{aligned}$$

Line current= $9.9A\angle 45^o$

Example 1.17 *In a parallel circuit, branch currents are given by $i_1 = 20\sqrt{2}sin(100\pi t + 60^o)A$, $i_2 = 20\sqrt{2}sin(100\pi t - 60^o)A$ and $i_3 = 20\sqrt{2}sin(100\pi t + 0^o)A$. What will be the r.m.s. value of the line current?*

Solution Given values:

$$\begin{aligned}i_1 &= 20\sqrt{2}sin(100\pi t + 60^o)A \\ i_2 &= 20\sqrt{2}sin(100\pi t - 60^o)A \\ i_3 &= 20\sqrt{2}sin(100\pi t + 0^o)A\end{aligned}$$

$$\begin{aligned}I_1 &= 20A\angle 60^o = (10 + \jmath 10\sqrt{3})A \\ I_2 &= 20A\angle - 60^o = (10 - \jmath 10\sqrt{3})A \\ I_3 &= 20A\angle 0^o = (20 + \jmath 0)A \\ I &= I_1 + I_2 + I_3 = (10 + 10 + 20) + \jmath(10\sqrt{3} - 10\sqrt{3} + 0) = 40A\angle 0^o\end{aligned}$$

Line current= $40A\angle 0^o$

Phasor Subtraction Consider two phasors A and B expressed as:

$$\begin{aligned}\overline{A} &= a_x + \jmath a_y \\ \overline{B} &= b_x + \jmath b_y \\ \overline{R} &= r_x + \jmath r_y = \bar{A} - \bar{B} = (a_x - b_x) + \jmath(a_y - b_y)\end{aligned}$$

ChapterAC/picACfundamentals/picPhasorSub.png

Figure 1.9: Phasor subtraction

Phasor Multiplication Consider two phasors A and B expressed as:

$$\begin{aligned}\overline{A} &= A\angle\phi_a \\ \overline{B} &= B\angle\phi_b \\ \overline{R} &= R\angle\phi_r = \bar{A}.\bar{B} = A\angle\phi_a B\angle\phi_b = A.B\angle\left(\phi_a + \phi_b\right)\end{aligned}$$

Phasor Division Consider two phasors A and B expressed as:

$$\begin{aligned}
\overline{A} &= A\angle\phi_a \\
\overline{B} &= B\angle\phi_b \\
\overline{R} &= R\angle\phi_r = \frac{\bar{A}}{\bar{B}} = \frac{A\angle\phi_a}{B\angle\phi_b} = \frac{A}{B}\angle\left(\phi_a - \phi_b\right)
\end{aligned}$$

Example 1.18 *Two phasors are given as follows: $A = 8 - \jmath 15$; $B = 4 + \jmath 6$. Find (a) $C = A + B$ and (b) $D = A - B$.*

Solution

$$\begin{aligned}
A &= 8 - \jmath 15 \\
B &= 4 + \jmath 6 \\
C &= A + B = 12 - \jmath 9 = 15\angle - 36.8699^o \\
D &= A - B = 4 - \jmath 21 = 21.3776\angle - 79.2157^o
\end{aligned}$$

Example 1.19 *Three phasors are given as follows: $A = 6 + \jmath 8$; $B = 5 + \jmath 7$; $C = 9 - \jmath 10$. Find (a) $X = \frac{A+B}{A+C}$ and (b) $\frac{CA}{A+B}$.*

Solution

$$\begin{aligned}
A &= 6 + \jmath 8 \\
B &= 5 + \jmath 7 \\
C &= 9 - \jmath 10 \\
A + B &= 11 + \jmath 15 = 18.6011\angle 53.7462^o \\
A + C &= 15 - \jmath 2 = 15.1327\angle - 7.5946^o \\
X &= \frac{A + B}{A + C} = 1.2292\angle 61.3408^o \\
CA &= 124 + \jmath 12 = 134.5362\angle 5.1173^o \\
Y &= \frac{AC}{A + B} = 7.2327\angle - 48.6288^o
\end{aligned}$$

1.4.5 Addition of waveforms

Addition of two or more waveforms can be obtained in any of the following three ways:

1. using plotting of waveforms.
2. using trigonometrical identities.
3. using phasors.

Consider two waveforms y_1, y_2 and their resultant y_r, expressed as:

$$\begin{aligned}
y_1 &= Y_{m1}\sin\omega t \\
y_2 &= Y_{m2}\sin(\omega t + \phi) \\
y_r &= Y_{mr}\sin(\omega t + \phi_r)
\end{aligned}$$

Using plotting of waveforms This is cumbersome tedious method to get the resultant. First we have to plot the waveforms of the sinusoids given and then adding point to point we obtain the resultant (Figure **??**).

ChapterAC/picACfundamentals/picAddWaves.png

Figure 1.10: Addition of two Sinusoidal waves

Using trigonometrical identities

$$\begin{aligned} y_r &= y_1 + y_2 = Y_{m1}\sin\omega t + Y_{m2}\sin(\omega t + \phi) \\ &= Y_{m1}\sin\omega t + Y_{m2}(\sin\omega t\cos\phi + \cos\omega t\sin\phi) \\ &= (Y_{m1} + Y_{m2}\cos\phi)\sin\omega t + Y_{m2}\cos\omega t\sin\phi \end{aligned}$$

Let $Y_{m1} + Y_{m2}\cos\phi = a$ and $Y_{m2}\sin\phi = b$.

$$\begin{aligned} y_r &= a\sin\omega t + b\cos\omega t \\ &= \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin\omega t + \frac{b}{\sqrt{a^2+b^2}}\cos\omega t\right] \end{aligned}$$

Let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi_r$ and $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi_r$.

$$\begin{aligned} y_r &= \sqrt{a^2+b^2}\,[\cos\phi_r\sin\omega t + \sin\phi_r\cos\omega t] \\ &= Y_{mr}\sin(\omega t + \phi_r) \end{aligned}$$

where $Y_{mr} = \sqrt{a^2+b^2}$ and $\phi_r = \tan^{-1}\frac{b}{a}$.

Using Phasors

$$\begin{aligned} \overline{Y_1} &= Y_1\angle 0^o = Y_1(1 + \jmath 0) \\ \overline{Y_2} &= Y_2\angle + \phi = Y_2(\cos\phi + \jmath\sin\phi) \\ \overline{Y_r} &= Y_r\angle\phi_r = \overline{Y_1} + \overline{Y_2} = (Y_1 + Y_2\cos\phi) + \jmath Y_2\sin\phi \end{aligned}$$

Hence, $Y_r = \sqrt{(Y_1 + Y_2\cos\phi)^2 + (Y_2\sin\phi)^2}$ and $\phi_r = \frac{Y_2\sin\phi}{Y_1 + Y_2\cos\phi}$

1.5 Power and power factor

Consider an ac source supplying a load shown in Figure 1.3a. Let v be the voltage applied alternating at a frequency f and i be the current flowing into the circuit. If the current lags the voltage by an angle ϕ, they may be expressed as:

$$\begin{aligned} v &= V_m\sin\omega t \\ i &= i_m\sin(\omega t - \phi) \end{aligned}$$

Instantaneous power, p consumed by the load is the product of instantaneous voltage and current. Figure 1.3b

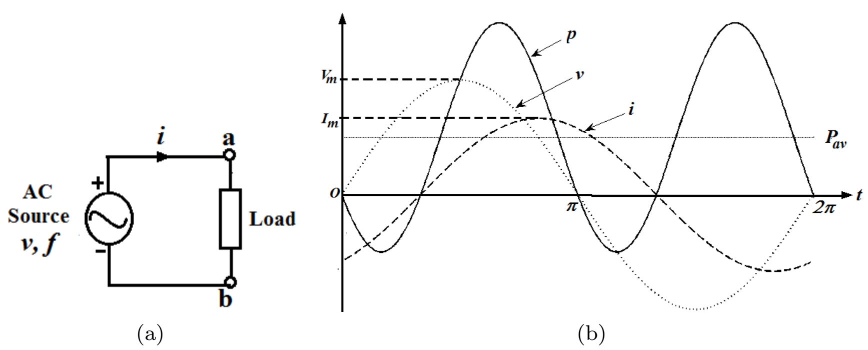


Figure 1.11: AC source connected to a load

shows v, i and p waveforms.

$$\begin{aligned} p = vi &= V_m\sin\omega t . I_m\sin(\omega t - \phi) \\ &= \frac{V_m I_m}{2}[\cos\phi - \cos(2\omega t - \phi)] \\ &= VI\,[\cos\phi - \cos(2\omega t - \phi)] \end{aligned}$$

Points to Observe

- The instantaneous power waveform alternates at a frequency $2f$ whereas both current and voltage alternate at a frequency f.
- The instantaneous power is sometimes positive and sometimes negative. This indicates sometimes power is consumed by the load and sometimes it returns power to the supply.

Average power: The average power consumed by the load

$$P_{av} = \frac{1}{T}\int_0^T pdt = \frac{VI}{T}\int_0^T [\cos\phi - \cos(2\omega t - \phi)]\,dt = VI\cos\phi$$

The product of supply voltage and current VI is called the apparent power . It is expressed in volt-amperes. It differs from the average power by a factor $\cos\phi$. This factor is known as **power factor**.

Points to remember:

- The magnitude of power factor varies from 0 to 1.
- If the current lags the voltage, the power factor is called as lagging power factor and assigned with a positive sign.
- If the current leads the voltage, the power factor is called as leading power factor and assigned with a negative sign.

Example 1.20 *In an ac circuit instantaneous voltage and current are given as:*

$$\begin{aligned} v &= 50\sin\omega t \;\; V \\ i &= 5\sin(\omega t - \frac{\pi}{3}) \;\; A \end{aligned}$$

What will be the value of average power consumed by this circuit?

Solution Average power consumed by the circuit

$$P = VI\cos\phi = \frac{50}{\sqrt{2}} \times \frac{5}{\sqrt{2}} \times cos\frac{\pi}{3} = 62.5W$$

Example 1.21 *An ac single phase motor takes 20A at 0.8pf lagging when connected to a 240V, 50Hz supply. What is the power taken by the motor?*

Solution Given values:

$$\begin{aligned} V &= 240Volts \\ f &= 50Hz \\ I &= 20A \\ \cos\phi &= 0.8(lag) \end{aligned}$$

Power taken by the motor, $P_{in} = VI\cos\phi = 240 \times 20 \times 0.8 = 3840W$

1.6 Response of Circuit Elements when connected to AC Supply

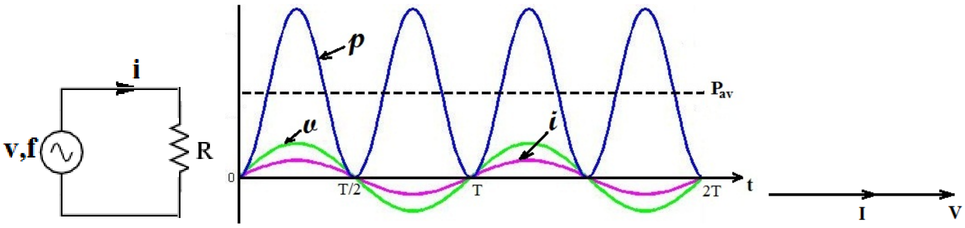
AC circuits comprise of resistance, inductance and capacitance. Before we study ac circuits, we shall first study how the individual components react when they are connected across ac source. Even though there is no pure components in reality, we shall consider ideal cases assuming that their other properties are negligible.

1.6.1 Purely Resistive Circuit

If a purely resistive circuit is excited by an alternating voltage source $v(t) = V_m \sin\omega t$ (Figure 1.4a), according to Ohm's law

$$i(t) = \frac{v(t)}{R} = \frac{V_m}{R}\sin\omega t = I_m \sin\omega t$$

The above equation shows that current flowing through the resistance is in phase with the applied voltage. Waveforms and phasor diagram are shown in Figures 1.4b and 1.4c respectively.



(a) Circuit (b) Waveforms (c) Phasor diagram

Figure 1.12: Purely resistive circuit

Reciprocal of resistance is known as **conductance** G, expressed in mho or Siemens. This may be defined as the ease provided by a pure resistor to an alternating (or direct) current passing through it.

Instantaneous power

$$p(t) = vi = V_m \sin\omega t I_m \sin\omega t = V_m I_m \sin^2\omega t = \frac{1}{2} V_m I_m (1-\cos 2\omega t) = VI(1-\cos 2\omega t)$$

I may be observed that $p(t)$ has two components. One is constant part and another varies with time at a frequency which is twice the supply frequency.

Average power

$$P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{VI}{T}\int_0^T (1-\cos 2\omega t)dt = VI$$

Energy input in one cycle

$$W = \int_0^T p(t)dt = VIT = I^2RT = I^2R/f$$

Example 1.22 *The voltage and current in an element are* $v = 200\sin(314t - 30^o)V$ *and* $i = 40\sin(314t - 30^o)V$. *Identify the element and its value.*

Solution Given $v = 200\sin(314t - 30^o)V$

$i = 40\sin(314t - 30^o)V$

It is observed that voltage and current are in same phase. Hence, this is a resistive circuit.

$R = \frac{V_m}{I_m} = \frac{200}{40} = 5\Omega$

Example 1.23 *A* 100Ω *resistance is carrying a sinusoidal current given by* $3\cos\omega t$. *Determine (i) instantaneous power taken by the resistance, (ii) average power.*
[U.P. Technical University,Electrical Engineering, First semester, 2003-04]

Solution (i) Instantaneous power taken by the resistance,

$$p = i^2R = (3\cos\omega t)^2 \times 100 = 900\cos^2\omega t = 450(1+\cos 2\omega t)$$

(ii) Average power

$$P_{av} = \frac{1}{\pi}\int_0^\pi p d\theta = 450W$$

1.6.2 Purely Inductive Circuit

A iron-cored coil made with thick copper wire with negligible resistance may be considered as a pure inductor. If a pure inductor is excited by an alternating voltage source $v(t) = V_m \sin\omega t$ (Figure 1.5a),

$$v(t) = L\frac{di}{dt}$$

$$di = \frac{V_m}{L}\sin\omega t dt$$

Integrating both sides

$$i = -\frac{V_m}{\omega L}\cos\omega t = I_m \sin(\omega t - 90^o) \quad (1.3)$$

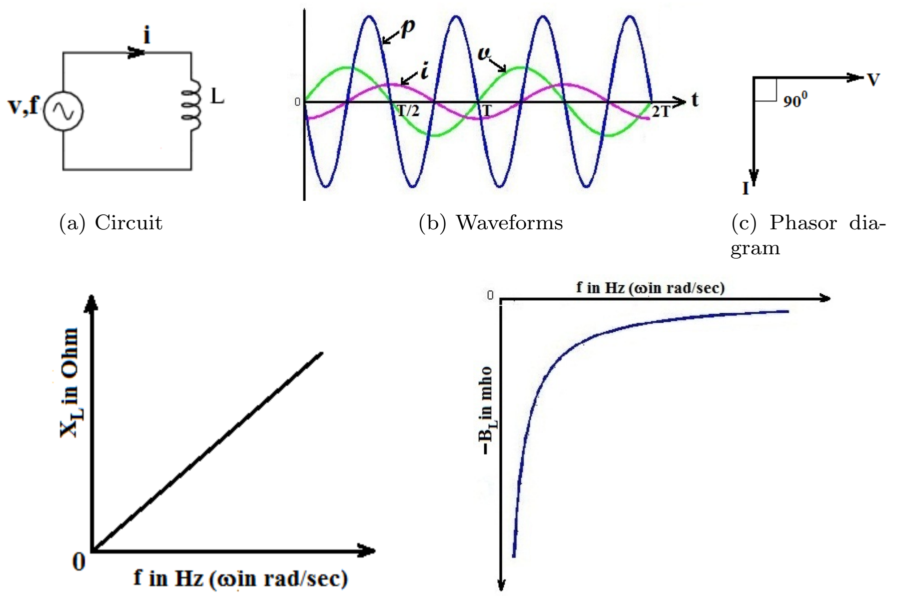
where $I_m = \frac{V_m}{X_L}$ and $X_L = \omega L$, known as inductive reactance.

$$[X_L] = \frac{[V_m]}{[I_m]} = \frac{volts}{amps} = ohm$$

Inductive reactance is a function of frequency as shown in Figure 1.5d. It becomes zero when $f = 0$. That is why inductor behaves as a short circuit when connected across d.c. supply.

Reciprocal of inductive reactance is known as inductive succeptance B_L, expressed in mho or Siemens. This may be defined as the ease provided by a pure inductor to an alternating current passing through it. B_L is again a funtion of frequency as shown in Figure 1.5e. It decreases as the frequency increases.

Equation 1.1 shows that voltage leads the current by 90^o. In other words, current lags the voltage by 90^o as shown in Figure 1.5b. Waveforms and phasor diagram are shown in Figures 1.5b and 1.5c respectively.



(a) Circuit (b) Waveforms (c) Phasor diagram

(d) Variation of X_L with frequency (e) Variation of B_L with frequency

Figure 1.13: Purely inductive circuit

Instantaneous power

$$p(t) = vi = -V_m \cos\omega t I_m \sin\omega t = -\frac{1}{2}V_m I_m \sin 2\omega t = VI\sin 2\omega t$$

Average power

$$P_{av} = \frac{1}{T}\int_0^T p(t)dt = -\frac{VI}{T}\int_0^T \sin 2\omega t dt = 0$$

Energy stored

$$W_s = \frac{1}{2}Li^2 = \frac{1}{2}I_m^2 \sin^2\omega t$$

Using phasors, equation 1.1 may be written as $\overline{V} = \overline{I}.\overline{X_L} = \jmath X_L \overline{I}$. Phasor diagram is shown in Figure 1.5c.

Example 1.24 *The voltage across the 15mH inductor is expressed as* $v = 15\sin(100t + 30^o)$. *What will be the expression of the circuit current?*

Solution

$$\begin{aligned}
V_m &= 150 volts \\
\omega &= 1100 rad/sec \\
L &= 15mH \\
X_L &= 1\omega L = 100 \times 15 \times 10^{-3} = 1.5\Omega \\
I_m &= 1\frac{V_m}{X_L} = \frac{15}{1.5} = 10A \\
i &= 110\sin(100t + 30^o - 90^o) = 10\sin(100t - 60^o)
\end{aligned}$$

Figure 1.14: Equivalent circuit of practical inductor

Practical inductors and Quality factor

Practically no pure inductors are available. Each coil has some internal resistance which is taken care of by connecting a resistance, R in series with L as shown in Fig. 1.6. The effectiveness of a coil as energy storing element is measured by a figure of merit, known as *quality factor, Q*. It is defined as

$$Q = 2\pi \frac{\text{maximum energy stored per cycle}}{\text{energy dissipated per cycle}}$$

Maximum energy stored per cycle=$\frac{1}{2}LI_m^2 = LI^2$

Energy dissipated per cycle=$I^2R/f = I^2R.\frac{2\pi}{\omega}$
As per definition

$$Q = 2\pi \frac{\text{maximum energy stored per cycle}}{\text{energy dissipated per cycle}} = 2\pi \frac{LI^2}{I^2R.\frac{2\pi}{\omega}} = \frac{\omega L}{R} = \frac{X_L}{R}$$

Hence, quality factor of a coil may be defined as the ratio of absolute value of reactance and its internal resistance. Smaller the internal resistance better the quality of a coil.

1.6.3 Purely Capacitive Circuit

If a purely capacitive circuit is excited by an alternating voltage source $v = V_m \sin \omega t$,

$$i(t) = C\frac{dv}{dt} = CV_m\omega \cos \omega t = \frac{V_m}{X_C} \sin(\omega t + 90^o) \tag{1.4}$$

where $I_m = \frac{V_m}{X_C}$ and $X_C = \frac{1}{\omega C}$, known as capacitive reactance.

$$[X_C] = \frac{[V_m]}{[I_m]} = \frac{volts}{amps} = ohm$$

Capacitive reactance is a function of frequency as shown in Figure 1.7d. It becomes infinity when $f = 0$. That is why capacitor behaves as an open circuit when connected across d.c. supply.

Reciprocal of capacitive reactance is known as capacitive succeptance B_C, expressed in mho or Siemens. This may be defined as the ease provided by a pure capacitor to an alternating current passing through it. B_C is again a funtion of frequency as shown in Figure 1.5e. It increases as the frequency increases.
Equation 1.2 shows that voltage lags the current by 90^o. In other words, current leads the voltage by 90^o. Waveforms and phasor diagram are shown in Figures 1.7b and 1.7c respectively.

$$p(t) = vi = V_m \sin \omega t I_m \cos \omega t = \frac{1}{2}V_m I_m \sin 2\omega t = VI \sin 2\omega t$$

Average power

$$P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{VI}{T}\int_0^T \sin 2\omega t dt = 0$$

Using phasors equation 1.2 may be written as $\overline{V} = \overline{I}.\overline{X_C} = -\jmath X_C.\overline{I}$. Phasor diagram is shown in Figure 1.7c.

Example 1.25 *The voltage and current in an element are* $v = 200\sin(314t - 75^o)V$ *and* $i = 25\sin(314t + 15^o)V$. *Identify the element and its value.*

Solution Given

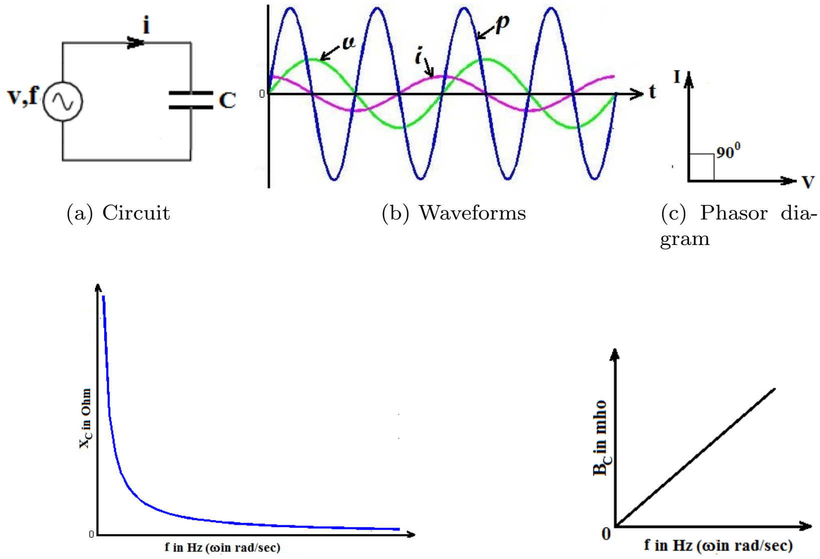
$v = 200\sin(314t - 75^o)V$

$i = 25\sin(314t + 15^o)V$.

Circuit current leads the voltage by $(75 + 15)^o = 90^o$. The circuit is capacitive.

$$X_C = \frac{200}{25} = 8\Omega$$

Hence, capacitance, $C = \frac{1}{\omega X_C} = \frac{1}{314 \times 8} = 398.09\mu F$



(a) Circuit (b) Waveforms (c) Phasor diagram

(d) Variation of X_C with frequency (e) Variation of B_C with frequency

Figure 1.15: Purely capacitive circuit

Example 1.26 *What is the reactance of* $20\mu F$ *capacitance at* $f_1 = 50Hz$, $f_2 = 100Hz$ *and* $f_3 = 400Hz$ *?*

Solution

At 50Hz capacitive reactance $X_{C1} = \frac{1}{2\pi f_1 C} = \frac{10^6}{2\pi\times 50\times 20} = 159.1549\Omega$

At 100Hz capacitive reactance $X_{C2} = \frac{1}{2\pi f_2 C} = \frac{10^6}{2\pi\times 100\times 20} = 79.5775\Omega$

At 400Hz capacitive reactance $X_{C3} = \frac{1}{2\pi f_3 C} = \frac{10^6}{2\pi\times 400\times 20} = 19.8944\Omega$

Note: With the increase in frequency, the capacitive reactance reduces.

Example 1.27 *At which frequency will a* $50\mu F$ *capacitor offers a reactance of* 100Ω*?*

Solution

$$\omega = \frac{1}{CX_C} = \frac{10^6}{50\times 100} = 200rad/sec$$

Frequency, $f = \frac{\omega}{2\pi} = 31.831Hz$

Practical capacitors and the Quality factor

Practically no pure capacitors are available. Each capacitor has some dielectric losses which is taken care of by connecting a resistance, R_p in parallel with C as shown in Fig. 1.8a. The effectiveness of a capacitor as energy storing element is measured by *quality factor, Q*.

Maximum energy stored per cycle=$\frac{1}{2}CV_m^2 = CV^2$

Energy dissipated per cycle=$I^2R/f = \frac{V^2}{R_p}.\frac{2\pi}{\omega}$

As per definition

$$Q = 2\pi\frac{\text{maximum energy stored per cycle}}{\text{energy dissipated per cycle}} = 2\pi\frac{CV^2}{\frac{V^2}{R_p}.\frac{2\pi}{\omega}} = \omega CR_p$$

Usually dielectric losses in a capacitor is very small. Thus R_p has a large value. Hence, quality factor of a capacitor is very high.

In practice, capacitor with smaller dielectric losses is represented by a series circuit as shown in Fig.1.8b. R_s being extremely small, $I_m \approx \frac{V_m}{X_C} = \omega CV_m$.

Maximum energy stored may be written as

$$W_s = \frac{1}{2}CV_m^2 = \frac{1}{2}\frac{I_m^2}{\omega^2 C} = \frac{I^2}{\omega^2 C}$$

(a) (b)

Figure 1.16: Equivalent circuits of practical capacitor

Energy dissipated per cycle

$$W_d = \frac{I^2 R_s}{f} = I^2 R_s . \frac{2\pi}{\omega}$$

As per definition

$$Q = 2\pi \frac{maximum\ energy\ stored\ per\ cycle}{energy\ dissipated\ per\ cycle} = 2\pi \frac{W_s}{W_d} = 2\pi \frac{\frac{I^2}{\omega^2 C}}{I^2 R_s \frac{2\pi}{\omega}} = \frac{1}{\omega C R_s} = \frac{X_C}{R_s}$$

Quality factor may be defined as the ratio of absolute value of reactance to its internal resistance.

In a nutshell

Frequency The number of cycles completed in one cycle is called the frequency, f. The unit of frequency is hertz (Hz).

Average value It is defined as the algebraic sum of all the instantaneous values divided by the number of values.

The average value of a sinusoidal current $I_{av} = 0.637 I_m$

Effective value or R.M.S. value The effective value of an alternating current is defined as the equivalent dc current which produces the same amount of heat when passed through a resistor for the same time duration. It is obtained from the root mean square values of a wave over a cycle. That's why it is also called the *root mean square value.*

The rms value of a sinusoidal current $I_{rms} = \frac{I_m}{\sqrt{2}}$

Form Factor Form factor is defined as the ratio of the rms value to the average value. It is denoted by k_f.

$$k_f = \frac{rms\ value}{average\ value}$$

For a sinusoid: $k_f = 1.11$

Peak Factor Peak factor is defined as the ratio of the peak value to the rms value. It is denoted by k_p.

$$k_p = \frac{peak\ value}{rms\ value}$$

For a sinusoid: $k_p = 1.414$

Representation of phasors

Polar form: $\overline{A} = A\angle\phi$ **Cartesian form:** $\overline{A} = a_x + \jmath a_y$

Relationship between polar and cartesian form:

$a_x = A\cos\phi$ $A = \sqrt{a_x^2 + a_y^2}$

$a_y = A\sin\phi$ $\phi = \tan^{-1}\frac{a_y}{a_x}$

Purely resistive circuit: Voltage and current are in same phase. Average power $P_{av} = VI$.

Purely inductive circuit: Current lags the supply voltage by 90^o. Average power $P_{av} = 0$.

Purely capacitive circuit: Current leads the supply voltage by 90^o. Average power $P_{av} = 0$.