$$I_{YY} = 990.1 \times 10^4 + 2\left(\frac{10 \times 200^3}{12}\right)$$

$$= 0.23 \times 10^8 \text{ mm}^4$$

Since I_{YY} is less than I_{XX}, the column will tend to buckle about Y-Y axis.

Least moment of inertia of the column section,

$$I = I_{min} = I_{YY} = 0.23 \times 10^8 \text{ mm}^4$$

The column is fixed at both ends.

∴ Effective length,

$$L = \frac{l}{2} = \frac{3000}{2} = 1500 \text{ mm}$$

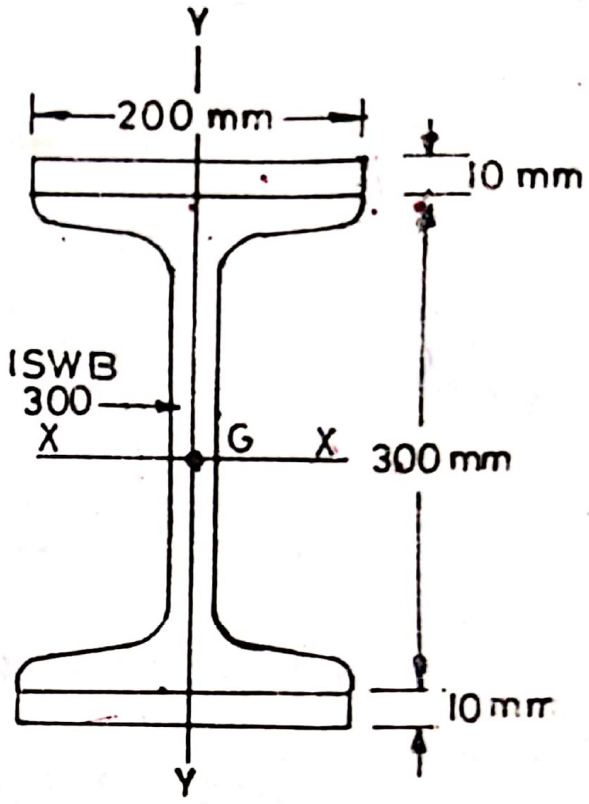

Fig. 19.10.

∴ Least radius of gyration of the column section,

$$K = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.23 \times 10^8}{10133}} = 47.64 \text{ mm}$$

Crippling load as given by Rankine's formula,

$$P_{cr} = \frac{f_c \times A}{1 + \alpha\left(\frac{L}{K}\right)^2} = \frac{320 \times 10133}{1 + \frac{1}{7500}\left(\frac{1500}{47.64}\right)^2} = 2864023.3 \text{ N}$$

$$\text{Safe load} = \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{2864023.3}{3} = 954674.43 \text{ N}$$ **Ans.**

Summary of results :

(*i*) When both ends of the column are pinned (hinged)

$$\text{Crippling load} = P = \frac{\pi^2 EI}{l^2}$$

(*ii*) When one end of the column is fixed and the other end is free,

$$\text{Crippling load} = P = \frac{\pi^2 EI}{4l^2}$$

(*iii*) When both ends of the column are fixed

$$\text{Crippling load} = P = 4\frac{\pi^2 EI}{l^2}$$

(*iv*) When one end of the column is fixed, and the other end is pinned (hinged),

$$\text{Crippling load} = P = \frac{2\pi^2 EI}{l^2}.$$

§3. EFFECTIVE LENGTH OF A COLUMN

The effective length of a given column with given end conditions is the length of an equivalent column of the same material and section with hinged ends having the value of the crippling load equal to that of the given column.

Actual length of column = l, effective length = L		
End conditions of column	*Crippling load*	*Relation between effective length and actual length*
Case 1 : Both ends hinged	$\frac{\pi^2 EI}{l^2} = \frac{\pi^2 EI}{L^2}$	$L = l$
Case 2 : One end fixed, one end free	$\frac{\pi^2 EI}{4l^2} = \frac{\pi^2 EI}{L^2}$	$L = 2l$
Case 3 : Both ends fixed	$\frac{4\pi^2 EI}{l^2} = \frac{\pi^2 EI}{L^2}$	$L = \frac{l}{2}$
Case 4 : One end fixed, one end hinged	$\frac{2\pi^2 EI}{l^2} = \frac{\pi^2 EI}{L^2}$	$L = \frac{l}{\sqrt{2}}$

If L be effective length of a column, the crippling load,

$$P = \frac{\pi^2 EI}{L^2}$$

The effective lengths of columns corresponding to different end conditions are given in the above table.

In the above formula I should be taken as the moment of inertia of the section about the axis of least resistance. Here I should be taken as least moment of inertia of the section.

Putting $I = AK^2$

where K = least radius of gyration of the column section.

We have, crippling load $= P = \frac{\pi^2 EAK^2}{L^2}$

or, the stress intensity corresponding to the crippling load $= \frac{P}{A} = \pi^2 E \frac{K^2}{L^2}$

$$\therefore \quad \frac{P}{A} = \frac{\pi^2 E}{\left(\frac{L}{K}\right)^2}$$

The ratio $\frac{L}{K} = \frac{\text{effective length}}{\text{least radius of gyration}}$ is called the *slenderness ratio* to the column.

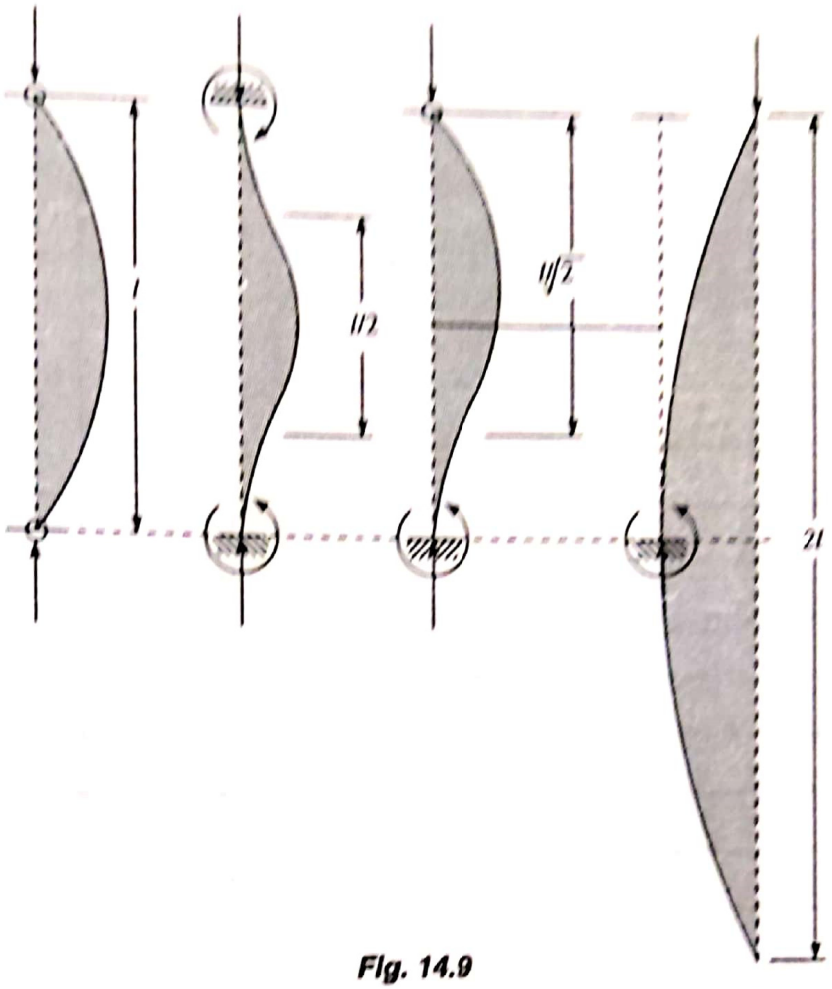

Fig. 14.9

§4. ASSUMPTIONS MADE IN EULER'S THEORY

Euler's formula for the crippling load is based on the following assumptions :

(*i*) The column is initially perfectly straight and is axially loaded.

(*ii*) The section of the column is uniform.

(*iii*) The column material is perfectly elastic, homogeneous and isotropic and obeys Hooke's law.

(*iv*) The length of the column is very large compared to the laternal dimensions.

(*v*) The direct stress is very small compared with the bending stress corresponding to the buckling condition.

(*vi*) The self-weight of the column is ignorable.

(*vii*) The column will fail by buckling alone.

§5. LIMITATION OF EULER'S FORMULA

The validity of Euler's formula is subject to the satisfaction of the assumptions mentioned in the above article. Further accepting the formula for the crippling load,

$$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 EAK^2}{L^2}$$

$$\text{The stress at failure} = \frac{P}{A} = P_e = \frac{\pi^2 E}{\left(\frac{L}{K}\right)^2}$$

We have, $$\frac{\pi^2 E}{\left(\frac{L}{K}\right)^2} = F_y$$

$$\left(\frac{L}{K}\right)^2 = \frac{\pi^2 E}{F_y} \quad \therefore \quad \frac{L}{K} = \pi\sqrt{\frac{E}{F_y}}$$

Taking $E = 2.1 \times 10^5\ N/mm^2$ and $F_y = 300\ N/mm^2$

$$\frac{L}{K} = \pi\sqrt{\frac{2.1\times10^5}{300}} = 83.1 \text{ Say } 80$$

General Comment: From Euler's equation critical load $P = \frac{\pi^2 EI}{L^2}$ we note that the critical load of a column is proportional to the flexural rigidity *EI* and inversely proportional to the sequare of the length. It is worthy to note that the strength of the material itself as represented by such property like proportional limit or yield stress does not find a place in the equation for the critical load. Thus, increasing a strength property does not increase the critical load of slender column. The critical load can be increased only by increasing the flexual rigidity, decreasing the length or providing additional lateral support.

Hence when the slenderness ratio is less than this limit for mild steel columns, Euler's formula will not be valid.

Problem 1. *A mild steel tube 4 metres long, 30 mm, internal diameter and 4 mm thick is used as a strut with both ends hinged. Find the collapsing load. Take $E = 2.1 \times 10^5\ N/mm^2$.*

Solution. Moment of inertia of section $= I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}[38^4 - 30^4] = 62592.6\ mm^4$.

Since both ends of the column are hinged

Effective length $= L = 4\ m. = 4000\ mm.$

$\therefore$ Crippling load $= P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 2.1\times10^5 \times 62592.6}{(4000)^2} = 8108\ N$

Problem 2. *A strut 2.50 metres long is 60 mm in diameter. One end of the strut is fixed while its other end is hinged. Find the safe compressive load for the member using Euler's formula, allowing a factor of safety of 3.5. Take $E = 2.1 \times 10^5\ N/mm^2$.*

Solution. Since one end of the strut is fixed and the other end is hinged, effective length of strut

$$= L = \frac{l}{\sqrt{2}} = \frac{2500}{\sqrt{2}} = 1768\ mm$$

Moment of inertia of the section $= I = \frac{\pi d^4}{64} = \frac{\pi\times60^4}{64} = 636172\ mm^4$.

$\therefore$ Crippling load $= P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2\times2.1\times10^5\times636172}{1768^2} = 421820\ N$

$\therefore$ Safe load $= \frac{\text{Crippling load}}{\text{Factor of safety}} = \frac{421820}{3.5} = 120520\ N.$

Problem 3. *A solid circular compression member 50 mm in diameter is to be replaced by a hollow circular section of the same material. Find the size of the hollow section if the internal diameter is 0.6 times the external diameter.*

SOLUTION. Diameter of the solid section = $D = 50\ mm$
External diameter of the hollow section = D_1,
Internal diameter of the hollow section = $D_2 = 0.6\ D_1$
For the two columns to have the same strength

$$I_{hollow} = I_{solid}$$

$$\frac{\pi}{64}\left[D_1^4 - 0.6^4 D_1^4\right] = \frac{\pi}{64} D^4$$

$$D_1^4 - 0.6^4 D_1^4 = D^4$$

$$D_1^4 = \frac{D^4}{1-0.6^4}$$

$$D_1^4 = \frac{D^4}{0.8704} = \frac{50^4}{0.8704}$$

$$\therefore \quad D_1 = 51.765\ mm$$

$$D_2 = 0.6 \times 51.765 = 31.059\ mm.$$

PROBLEM 4. *Find the shortest length L for a pin ended steel column having a cross-section of 60 mm × 100 mm for which Euler's formula applies. Take $E_s = 2 \times 10^5\ N/mm^2$ and critical proportional limit is 250 N/mm^2.*

SOLUTION. Least moment of inertia of the section = $\frac{100 \times 60^3}{12} = 18 \times 10^5\ mm^4$.

Euler's critical load $= P = \frac{\pi^3 EI}{L^2}$

Stress corresponding to Euler's critical load = $\frac{\pi^2 EI}{AL^2}$

When this stress reaches 250 N/mm^2.

$$\frac{\pi^2 EI}{AL^2} = 250$$

$$L^2 = \frac{\pi^2 EI}{A \times 250} = \frac{\pi^2 \times 2 \times 10^5 \times 18 \times 10^5}{(60 \times 100) \times 250} = 2.3687 \times 10^6$$

$$\therefore \quad L = 1539\ mm.$$

PROBLEM 5. *Calculate the critical load for a strut which is made of a bar circular in section and 5 metres long and which is pin-jointed at both ends. The same bar when freely supported gives a mid-span deflection of 10 mm under a load of 80 N at the centre.*

SOLUTION. *Analysis as a beam*
$l = 5000\ mm.$, $W = 80\ N$, $\delta = 10\ mm$,

$$\delta = \frac{Wl^3}{48EI} \quad \therefore \quad EI = \frac{Wl^3}{48\delta} = \frac{80 \times 5000^3}{48 \times 10} = \frac{125}{6} \times 10^9\ N/mm^2$$

Analysis as a strut

Euler's critical load

$$P = \frac{\pi^3 EI}{l^2} = \pi^3 \times \frac{125}{6} \times 10^9 \times \frac{1}{5000^2} = 8225\ N.$$

PROBLEM 6. *A bar 20 mm in diameter is 1.25 m long. When the bar is simply supported at its ends in a horizontal position and loaded with a concentrated load of 370 N the central deflection was found to be 9.50 mm.*

If the bar is placed vertically and loaded axially what would be the buckling load ? Find also the ratio of the maximum bending stress to the buckling stress.

PROBLEM 7. *Calculate the safe compressive load on a hollow cast iron column (one end rigidly fixed and the other hinged) of 150 mm external diameter and 100 mm internal diameter and 10 m in length. Use Euler's formula with a factor of safety of 5 and E = 95 K N/mm².*

SOLUTION.

$$D = 150\ mm \quad d = 100\ mm$$

$$L = \frac{l}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 5\sqrt{2m} = 5\sqrt{2} \times 1000\ mm. = 5000\sqrt{2}\ mm$$

$$I = \frac{\pi}{64}(150^4 - 100^4) = 1.994 \times 10^7\ mm^4$$

$$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 95 \times 1.994 \times 10^7}{(5000\sqrt{2})^2} = 373.92\ kN$$

$$\text{Safe load} = \frac{373.92}{5} = 74.78\ kN.$$

PROBLEM 8. *A hollow alloy tube 5 metre long with external and internal diameters equal to 40 mm and 25 mm respectively was found to extend by 6.4 mm under a tensile load of 60 kN. Find the buckling load for the tube, when used as a column with both ends pinned. Also find the safe compressive load for the tube, with a factor of safety of 4.*

SOLUTION. Area of the section $= A = \frac{\pi}{4}(40^2 - 25^2)\ mm^2 = 765.76\ mm^2$

When subjected to tension

$$\text{Extension } \delta = \frac{Wl}{AE}$$

$$\therefore \quad E = \frac{WL}{A\delta} = \frac{60 \times 10^3 \times 5 \times 10^3}{765.76 \times 6.4} = 61213.69\ N/mm^2$$

When used as a column

$$I = \frac{\pi}{64}(40^4 - 25^4) = 106489\ mm^4$$

Buckling load

$$P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 61213.69 \times 106489}{5000^2} = 2573.43\ N$$

Safe load, with a factor of safety of 4 $= \dfrac{2573.43}{4} = 643.34\ N.$

Problem 9. *A bar of length 4 metres when used as a simply supported beam and subjected to a uniformly distributed load of 30 kN/m over the whole span deflects 15 mm at the centre. Determine the crippling loads when it is used as a column with the following conditions :*

(i) Both ends pin jointed,

(ii) One end fixed and the other hinged,

(iii) Both ends fixed.

Solution. *Analysis as a beam*

$w = 30\ kN/m = 30\ N/mm.$

$l = 4000\ mm, \delta = 15\ mm.$

$$\delta = \frac{5}{384}\frac{wl^4}{EI}$$

$$\therefore \quad EI = \frac{5}{384}\cdot\frac{wl^4}{\delta} = \frac{5}{384}\times\frac{30\times(4000)^4}{15} = \frac{2}{3}\times 10^{13}\ Nmm^2.$$

Analysis as a column

(i) When both ends are hinged

Crippling load $$P = \frac{\pi^2 EI}{l^2} = \pi^2\times\frac{2}{3}\times 10^{13}\times\frac{1}{(4000)^2}$$

$$= 4112335\ N = 4112.335\ kN$$

(ii) When one end is fixed and the other end is hinged.

Crippling load $$P = \frac{2\pi^2 EI}{l^2} = 2\times 4112.335 = 8224.67\ kN$$

(iii) When both ends are fixed

Crippling load $$P = \frac{4\pi^2 EI}{l^2} = 4\times 4112.335 = 16449.34\ kN.$$

Problem 10. *A steel bar of rectangular section 30 mm × 40 mm pinned at each end is subjected to axial compression. The bar is 1.75 m long. Determine the buckling load and the corresponding axial stress using Euler's formula.*

Determine the minimum length for which Euler's equation may be used to determine the buckling load, if the proportional limit of the material is 200 N/mm². Take E = 2 × 10⁵ N/mm².

Solution. Least moment of inertia of the section of the bar $= I = \dfrac{40\times 30^3}{12} = 90000\ mm^4$

Since both ends are pinned,

Effective length $= L = l = 1750\ mm$

Euler's critical load $$= P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2\times 2\times 10^5\times 90000}{1750^2}\ N = 58009\ N$$

Corresponding axial stress $$= \frac{P}{A} = \frac{58009}{30\times 40}\ N/mm^2 = 48.34\ N/mm^2$$

We know by Euler's formula $$P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 EAK^2}{l^2}$$

$$\frac{P}{A} = \frac{\pi^2 EK^2}{l^2} = \frac{\pi^2 E}{\left(\dfrac{l}{K}\right)^2}$$

Problem 13. *A round steel rod of diameter 15 mm and length 2 metre is subjected to a gradually increasing axial compressive load. Using Euler's formula find the buckling load. Find also the maximum lateral deflection corresponding to the buckling condition. Both ends of the rod may be taken as hinged. Take* $E = 2.1 \times 10^5$ *N/mm*^{2} *and the yield stress of steel* $= 250$ *N/mm*^{2}.

Solution. Area of the rod

$$= A = \frac{\pi d^2}{4} = \frac{\pi}{4} \times 15^2 = 176.71 \text{ mm}^2.$$

Moment of inertia of the section $= I = \dfrac{\pi d^4}{64} = \dfrac{\pi \times 15^4}{64} = 2485.05 \text{ mm}^4.$

Since both ends of the member are hinged, the effective length

$$= L = 2000 \text{ mm}.$$

$\therefore$ Buckling load (using Euler's formula)

$$= P = \frac{\pi^2 \times 2.1 \times 10^5 \times 2485.05}{(2000)^2} N = 1287.6 \text{ N}$$

Direct compressive stress $= P_o = \dfrac{P}{A} = \dfrac{1287.6}{176.71} \text{ N/mm}^2 = 7.29 \text{ N/mm}^2.$

Let the maximum bending stress corresponding to buckling condition be p_b.

$\therefore$ $p_b + p_o =$ yield stress

$\therefore$ $p_b = 250 - 7.29 = 242.71 \text{ N/mm}^2.$

Let M be the maximum bending moment which occurs at the centre

$\therefore$ $$\frac{M}{I} = \frac{p_b}{\left(\frac{d}{2}\right)}$$

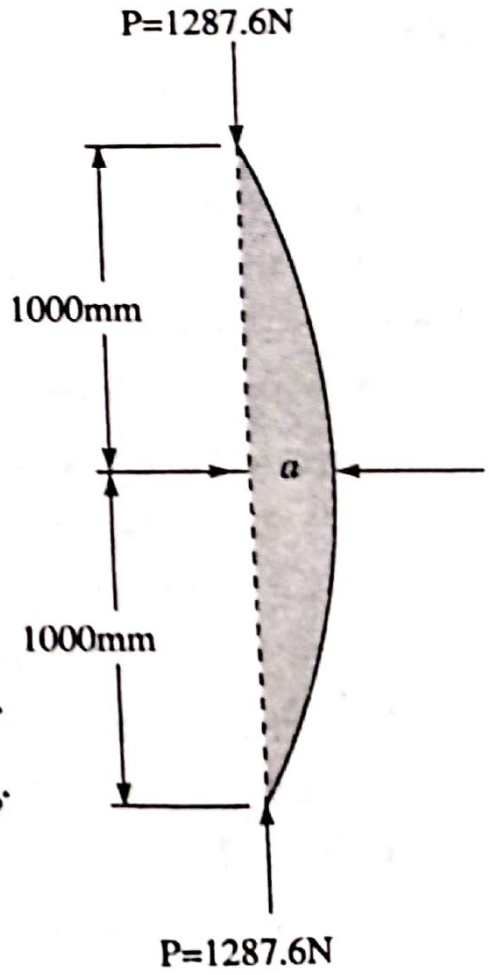

Fig. 14.11

$$\therefore \quad M = \frac{242.71}{7.5} \times 2485.05 \text{ Nmm.} = 80419.53 \text{ Nmm.}$$

Let the maximum central deflection be a mm.

$$\therefore \quad M = Pa = 1287.6\, a = 80419.53$$

$$\therefore \quad a = \frac{80419.53}{1287.6} = 62.46 \text{ mm.}$$

Example 19.4. A column having a T section with a flange 120 mm × 16 mm and web 150 mm × 16 mm is 3 m long. Assuming the column to be hinged at both ends, find the crippling load by using Euler's formula. $E = 2 \times 10^6$ kg/cm^2.

Solution : Given : Flange width = 120 mm = 12 cm

Flange thickness = 16 mm = 1.6 cm

Length of the web = 150 mm = 15 cm

Width of the web = 16 mm = 1.6 cm

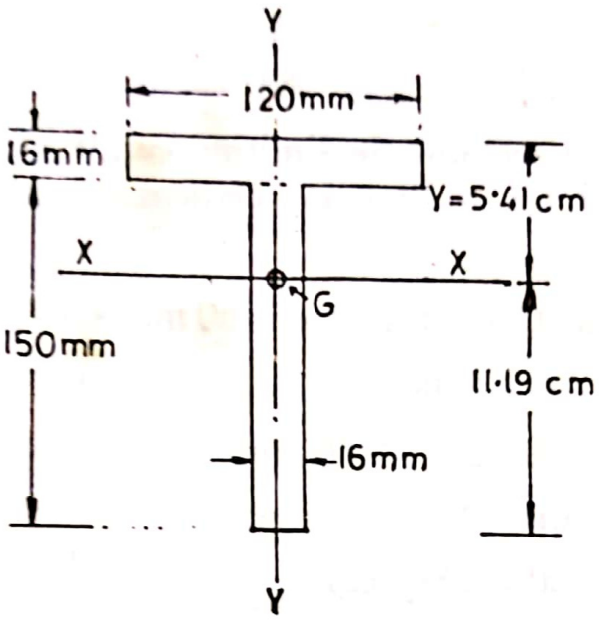

Fig. 19.7.

Young's modulus, $E = 2 \times 10^6$ kg/cm^2

Length of the column, $l = 3$ m $= 300$ cm

Since the column is hinged at both ends, effective length of the column, $L = l = 300$ cm

From Fig. 19.7, Y-Y is the axis of symmetry.

$\therefore$ The C.G. of the whole section lies on Y-Y axis.

Let the distance of the C.G. from the topmost fibre of the section = $\bar{Y}$

$$\therefore \quad \bar{Y} = \frac{12 \times 1.6 \times \frac{1.6}{2} + 15 \times 1.6\left(1.6 + \frac{15}{2}\right)}{12 \times 1.6 + 15 \times 1.6} = 5.41 \text{ cm}$$

Distance of C.G. from bottom fibre = (15 + 1.6) – 5.41 = 11.19 cm

Now M.I. of the whole section about X-X axis,

$$I_{XX} = \frac{12 \times 1.6^3}{12} + (12 \times 1.6)\left(5.41 - \frac{1.6}{2}\right)^2 + \frac{1.6 \times 15^3}{12}$$

$$+ (1.6 \times 15)\left(11.19 - \frac{15}{2}\right)^2 = 1188.92 \text{ cm}^4$$

M.I. of the whole section about Y-Y axis

Example 19.5. Determine the limiting length of a pin-ended column of section 60 mm × 100 mm so that the critical stress is 250 N/mm^2. Assume $E = 2 \times 10^5$ N/mm^2.

Solution : Given : Column section = 60 mm × 100 mm

Critical stress = 250 N/mm^2

Young's modulus $E = 2 \times 10^5$ N/mm^2

Sectional area of the column section

$$= 60 \times 100 = 6000 \text{ mm}^2$$

Critical load, P_{Cr} = Critical stress × sectional area

$$= 250 \times 6000 = 150 \times 10^4 \text{ N}$$

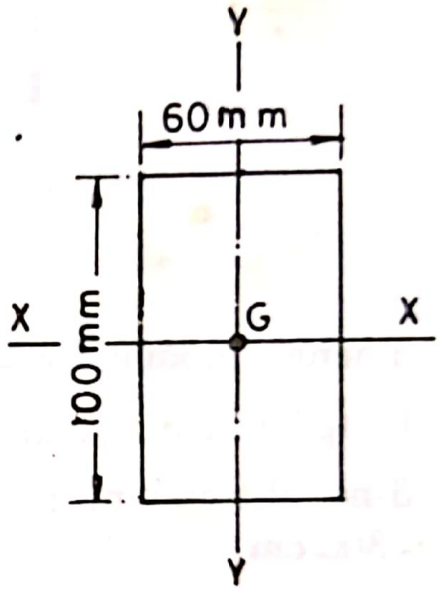

Fig. 19.8.

Least M.I. of the column section,

$$I = I_{min} = I_{YY} = \frac{100 \times 60^3}{12} = 180 \times 10^4 \text{ mm}^4$$

Since the ends of the column are pinned (*i.e.*, hinged),

$\therefore$ Effective length of the column L = Actual length of the column l

Applying Euler's formula,

$$P_{Cr} = \frac{\pi^2 \times EI}{L^2}$$

or $$150 \times 10^4 = \frac{\pi^2 \times 2 \times 10^5 \times 180 \times 10^4}{L^2}$$

or $$L^2 = \frac{\pi^2 \times 2 \times 10^5 \times 180 \times 10^4}{150 \times 10^4}$$

$\therefore$ $$L = 1539 \text{ mm} = 1.539 \text{ m}$$

$\therefore$ Required limiting length of the column = 1.539 m **Ans.**

Example 19.6. Calculate the Euler's critical load for a strut, square in section and 4 metres long with both ends hinged. The same bar is found to deflect 5 mm at the centre when a load of 100 N is placed at mid-span with the bar simply supported at the ends.

Solution : Given : Length of the strut, $l = 4$ m $= 4000$ mm

Central deflection of the bar, $\delta = 5$ mm

Point load at the centre of the bar, $W = 100$ N

We know central deflection under point load at the centre of the span when simply supported

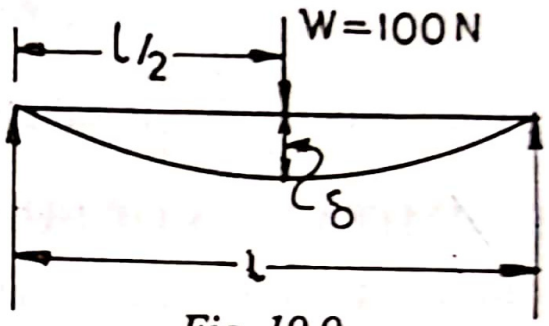

Fig. 19.9.

$$\delta = \frac{Wl^3}{48EI}$$

or $$5 = \frac{100 \times 4000^3}{48 \times EI}$$

or $$EI = \frac{100 \times 4000^3}{48 \times 5} = 2.67 \times 10^{10} \text{ Nmm}^2$$

$\because$ The strut is hinged at both ends,

Effective length of the strut $(L) = l = 4000$ mm

$\therefore$ Euler's critical load for the strut,

$$P_{Cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 2.67 \times 10^{10}}{4000^2} = 16469.90 \text{ N} = 16.47 \text{ kN (Approx)}$$ **Ans.**

19.14. SLENDERNESS RATIO

Example 19.11. A built up column consisting of rolled steel beam ISWB 300 with two plates 200 mm × 10 mm connected at the top and bottom flanges. Calculate the safe load the column can carry, if the length is 3 m and both ends are fixed. Take factor of safety 3. $f_c = 320$ N/mm^2 and $\alpha = \frac{1}{7500}$.

Take properties of joist : $A = 6133$ mm^2.

$I_{XX} = 9821.6 \times 10^4$ mm^4; $I_{YY} = 990.1 \times 10^4$ mm^4

Solution : Given : Length of the built up column, $l = 3$ m $= 3000$ mm

Factor of safety = 3

$$f_c = 320 \text{ N/mm}^2$$

$$\alpha = \frac{1}{7500}$$

Sectional area of the built up column,

$$A = 6133 + 2\,(200 \times 10) = 10133 \text{ mm}^2$$

Moment of inertia of the built up column section about XX axis,

$$I_{XX} = 9821.6 \times 10^4 + 2\left[\frac{200 \times 10^3}{12} + (200 \times 10)\,155^2\right] = 1.94 \times 10^8 \text{ mm}^4$$

Moment of inertia of the built up column section about YY axis,